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INHOMOGENEOUS DISTRIBUTION OF THE

RADIOACTIVE HEAT SOURCES.

I. Theory *

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by

Chi-yuen Wang

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INHOMOGENEOUS DISTRIBUTION OF THE RADIOACTIVE HEAT SOURCES¹

I. Theory

by

Chi-yuen Wang²

1. Introduction

The present data about heat flow over the surface of the earth have been analyzed by Wang (1963) and by Lee and MacDonald (1963). The analyses were based mainly on data summarized by Lee (1963).

Early in 1963 Mr. Joughin helped me to arrange the heat-flow data on IBM cards and plot them on a global map; I then averaged them over $10^\circ \times 10^\circ$ squares and $20^\circ \times 20^\circ$ squares. The averaging processes tend to bring down the amplitudes of the fluctuations indicated by individual data and to smooth out the values. The main results of the analyses are: (1) for the $10^\circ \times 10^\circ$ averages, there appears to be a strong correlation between the heat-flow values and some major geological structures; for example, very high heat-flow values (2.5×10^{-6} cal/cm² sec or above) tend to correlate with such areas as the western mountainous and plateau regions of North America, the East Pacific Rise, the Mid-Atlantic Ridge, and the island arc of the western Pacific Ocean and some other volcanic areas; (2) for the $20^\circ \times 20^\circ$ averages, there appear to be certain correlations between the highs and lows of the heat flow and the negative and positive geoid heights represented by the low-order spherical harmonics as obtained from satellite data (Izsak, 1963a); the coefficient of correlation was 0.5.

The implication of the second result is that under the depressed geoid the material may be hotter and lighter, one being related to the other, and, contrarily, under the elevated geoid the material may be cooler and heavier. I explained, qualitatively and tentatively, the correlated phenomena by the convection-current hypothesis, by which the ascending currents of lighter and hotter material bring up heat to the top of the mantle, and higher surface heat flow appears over the ascending currents while the gravity is lower than its surroundings.

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Lee and MacDonald (1963) used a larger amount of data (757 values) and represented the heat flow by contours. In dealing with the very unevenly distributed data, they employed functions which are orthogonal to the station net. Furthermore, they obtained the spherical harmonic coefficients, corresponding to the various representations, up to order 2. They also noticed a rough correlation between the heat-flow field and the gravitational field and suggested that convection current hypothesis may explain the phenomenon.

These observed surface thermal and gravitational phenomena do not, of course, necessarily indicate motions in the mantle; inhomogeneity in the physical properties of the upper mantle or in the material of the mantle itself may also be the possible cause for the correlation between the two sets of geophysical data. In this paper I assume that the inhomogeneous distribution of the heat sources in the interior is responsible, through thermal conduction, for the fluctuations of the surface heat flow. The interior temperature so related is assumed to cause uneven thermal expansion and to produce density anomaly, which in turn causes the gravity anomaly observed on the surface of the earth. The purpose of this paper is to form a theory to test whether the above assumptions are sound. Numerical results will be given in a later publication.

2. New correlation coefficients

The present correlation is based on two sources of data: (1) Izsak's gravitational potential from satellite orbits (1963b) and (2) Lee and MacDonald's heat-flow analysis (1963). The coefficients in the spherical harmonic expansion of the heat-flow analysis are listed in table 1. The expansion is expressed as

$$F(\theta, \varphi) = \sum_{\ell=0}^2 \sum_{m=0}^{\ell} (f_{\ell,m} \cos m\varphi + g_{\ell,m} \sin m\varphi) P_{\ell}^m(\cos \theta) .$$

Since the above harmonic expansion is only up to the second order, I choose, for the purpose of correlation, only the second-order tesseral harmonics. The geoid height according to the conventional coefficients $c_{2,2} = 7.56 \times 10^{-7}$ and $s_{2,2} = -6.15 \times 10^{-7}$ is plotted in figure 1.

Furthermore, to get a clearer view of the correlation, we are more interested in the variations in the heat-flow distribution than in the values of heat flow themselves. The main spherical term $f_{0,0}$ is therefore taken out and in figures 2, 3, and 4 the fluctuations of heat flow with respect to the references $f_{0,0}$ are plotted according to the coefficients listed in table 1. The correlation coefficients between figure 1 and figures 2, 3, and 4 are, respectively, -0.77, -0.70, -0.82.

Table 1

Coefficients in Spherical Harmonic Expansion in $\mu\text{cal}/\text{cm}^2\text{sec}$

(from Lee and MacDonald, 1963)

| | $5^\circ \times 5^\circ$ weighted averages | $45^\circ \times 45^\circ$ weighted averages | Extreme values deleted (611 values) |
|-----------|---|---|--|
| $f_{0,0}$ | 1.529 | 1.509 | 1.402 |
| $f_{1,0}$ | 0.039 | 0.152 | 0.025 |
| $f_{1,1}$ | 0.059 | 0.072 | -0.006 |
| $f_{2,0}$ | 0.057 | -0.062 | 0.058 |
| $f_{2,1}$ | -0.036 | 0.042 | -0.033 |
| $f_{2,2}$ | -0.075 | 0.029 | -0.052 |
| $g_{1,1}$ | 0.032 | 0.100 | 0.032 |
| $g_{2,1}$ | 0.085 | 0.113 | 0.048 |
| $g_{2,2}$ | 0.038 | 0.052 | 0.039 |

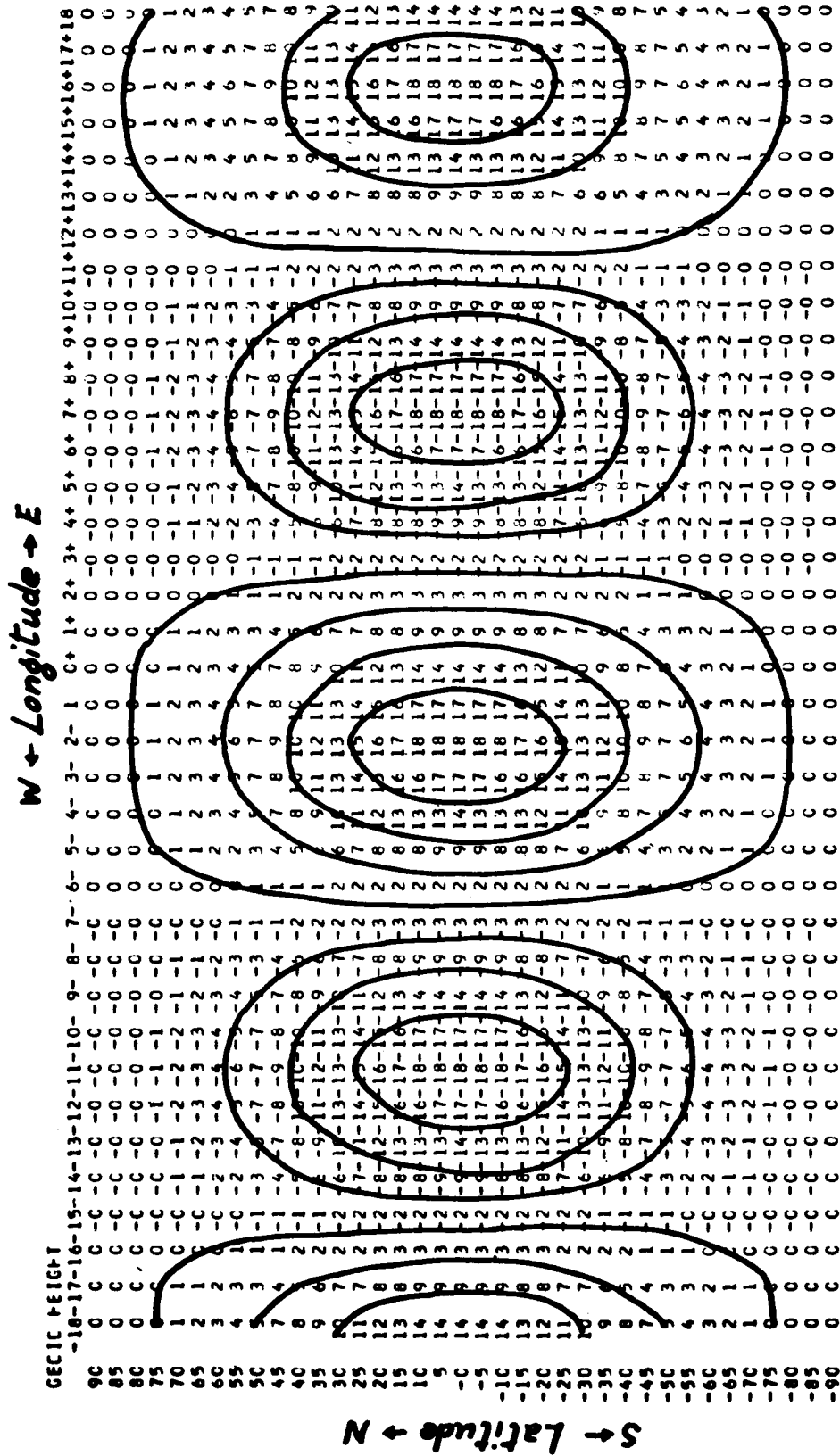


Figure 1--Elevations of the geoid in meters above the reference ellipsoid with flattening 1/298.24.
(Coefficients of the spherical harmonics in the geopotential are of the second order)

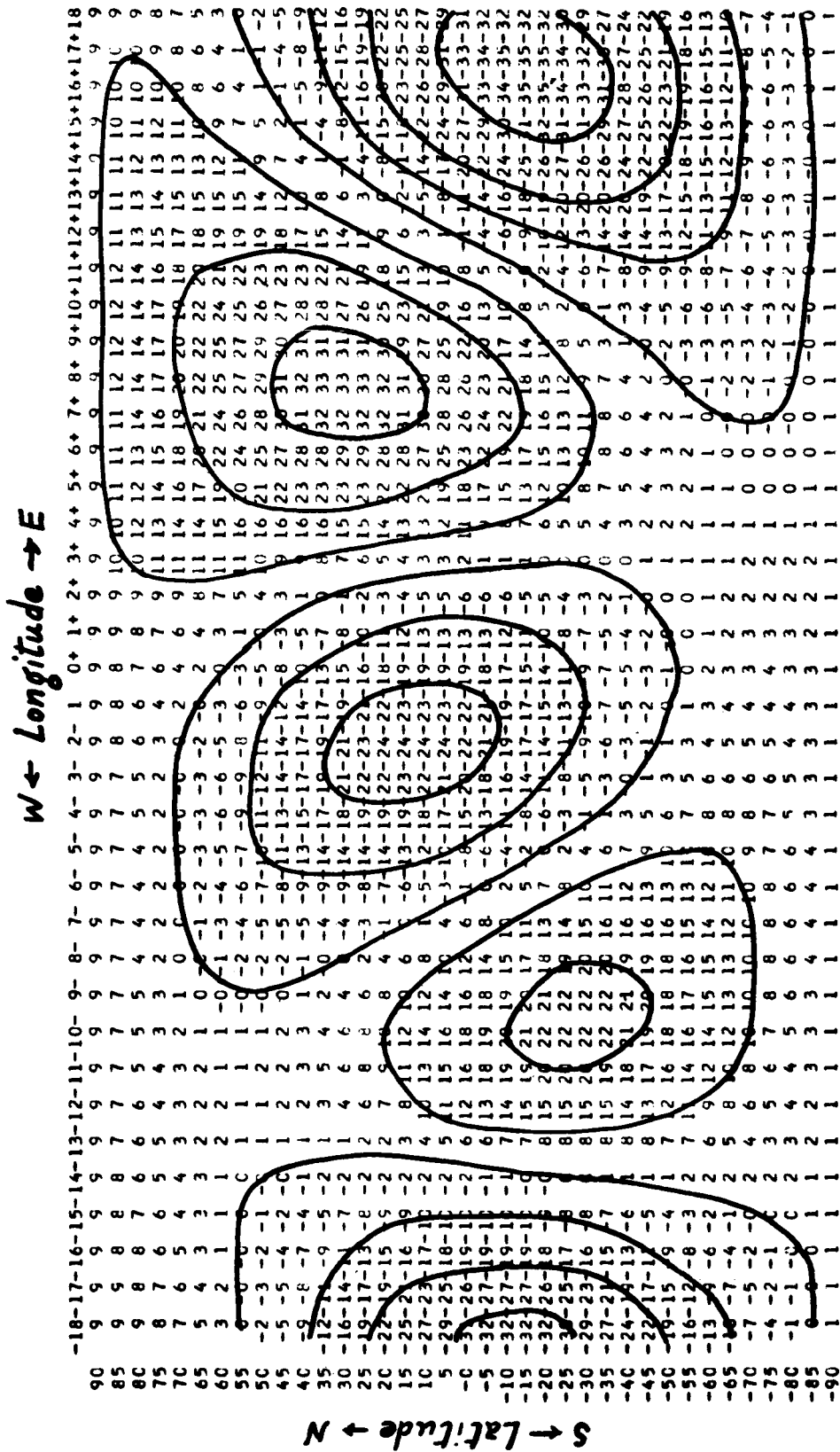


Figure 2.---Heat-flow fluctuations in $10^{-2} \mu\text{cal}/\text{cm}^2 \text{ sec}$, as referred to a mean heat-flow $1.51 \mu\text{cal}/\text{cm}^2 \text{ sec}$.
 $(5^\circ \times 5^\circ$ weighted averages corresponding to the left-hand column of table 1 without $f_{0,0})$

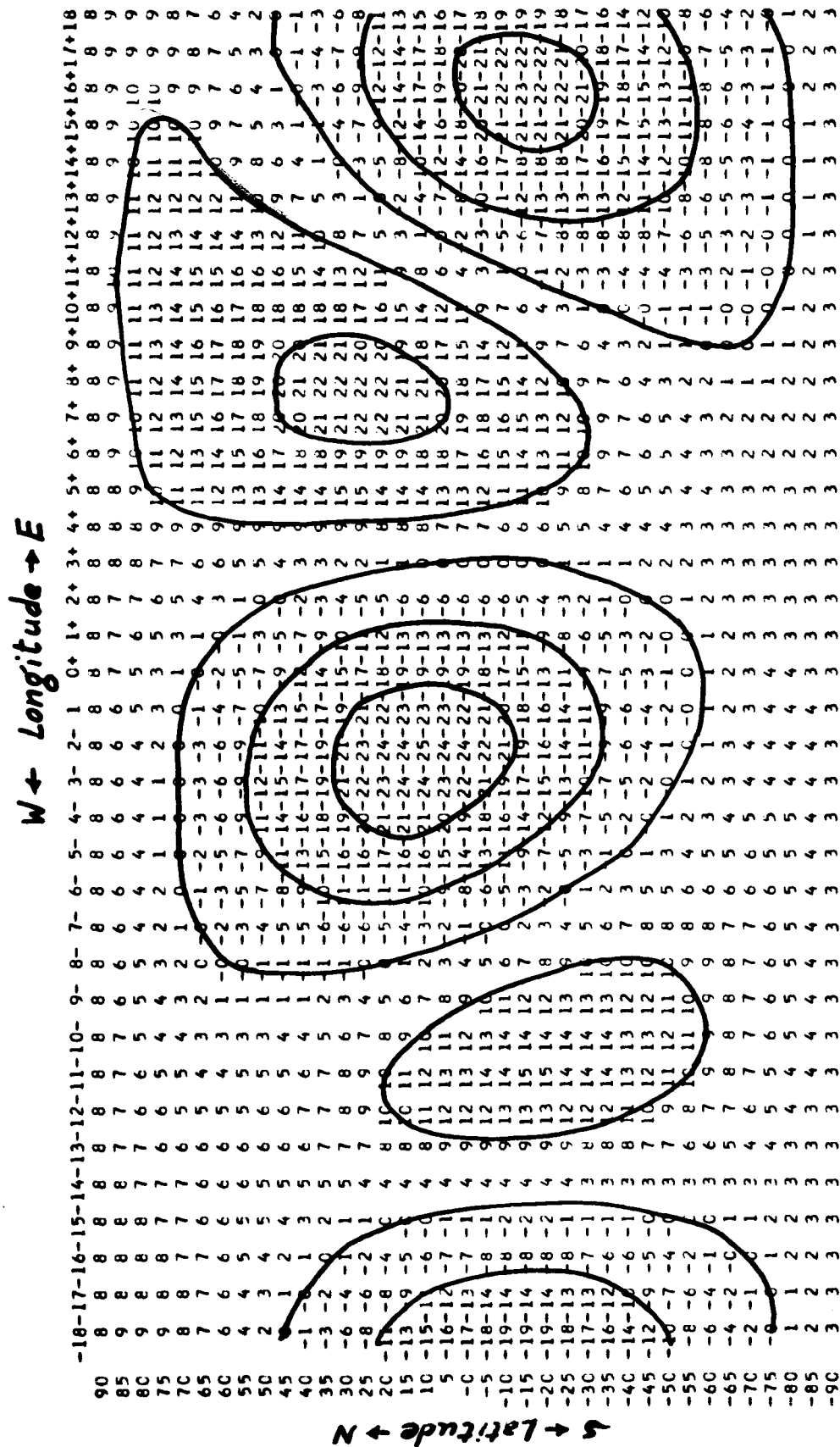


Figure 4.--Heat-flow fluctuations in $10^{-2} \text{ ucal/cm}^2 \text{ sec}$, as referred to a mean heat-flow $1.40 \cdot 10^{-2} \text{ ucal/cm}^2 \text{ sec}$ (extreme values deleted corresponding to the right-hand column of table 1 without $f_{o,o}$)

The large negative values of the linear correlation coefficient imply a close relation between the density perturbations and the temperature variations in the interior of the earth. The temperature variations may result from inhomogeneous distribution of radioactive sources in the interior; this will be discussed in the next section.

In a recent communication Dr. John A. O'Keefe pointed out that the nonhydrostatic components in the geopotential should be referred to the hydrostatic ellipsoid ($f = 1/299.8$) instead of to the best-fitting ellipsoid. However, geoid height plotted on the hydrostatic ellipsoid shows no correlation to the heat-flow distribution. I am tempted to think that the nonhydrostatic part of the second-order zonal harmonic in the geopotential, which is much larger than the other nonhydrostatic components, is mainly the result of some mechanism other than the temperature variations in the earth's interior.

3. Temperature resulting from inhomogeneous distribution of the radioactive heat sources

In this section I shall draw some models for the distribution of the radioactive heat sources in the interior that would give the fluctuations of surface heat flow through thermal conduction. Our present knowledge about the earth's interior is extremely limited. It is therefore necessary, for the time being, to treat the interior with some much-simplified models. In the present treatment, I have assumed that the thermal conductivity K , the thermal diffusivity k , and the thermal expansion coefficient α are constants throughout the interior.

The temperature T_m resulting from a distribution of heat sources generating heat at a rate A per unit volume is controlled by the conduction equation

$$k \nabla^2 T_m - \frac{\partial}{\partial t} T_m = \frac{k}{K} A = \psi_m, \quad (1)$$

where ψ_m is the source density function. Owing to the radioactive decay of the heat sources, ψ_m should be a decreasing function of time, but as uranium, thorium and potassium all have very long half-lives (between $0.7 \sim 4 \times 10^9$ years), the decrease in ψ_m is very slow and may be neglected here.

To transfer the above equation into dimensionless form, assume

$$\begin{aligned} T_m^0 &= \text{characteristic temperature} \\ t^0 &= \text{time} \\ r^0 &= \text{length} \\ \psi_m^0 &= \text{source density,} \end{aligned}$$

and assume

$$T_m = T_m^0 T_m^*, \quad t = t^0 t^*, \quad r = r^0 r^* \quad \text{and} \quad \psi_m = \psi_m^0 \psi_m^* . \quad (2)$$

Equation (1) then becomes

$$\left(\frac{k}{r^{02}} T_m^0 \right) \nabla^{*2} T_m^* - \left(\frac{T_m^0}{t^0} \right) \frac{\partial T_m^*}{\partial t^*} = \psi_m^0 \psi_m^*$$

or

$$\nabla^{*2} T_m^* - \alpha \frac{\partial T_m^*}{\partial t^*} = \beta \psi_m^* , \quad (3)$$

where

$$\alpha = \frac{r^{02}}{k t^0}$$

and

$$\beta = \frac{r^{02} \psi_m^0}{k T_m^0} .$$

If $\alpha = 1$ and $\beta = 1$, the above equations give

$$r^0 = \sqrt{k t^0} \quad (4)$$

and

$$\psi_m^0 = T_m^0 / t^0 , \quad (5)$$

and we arrive at the dimensionless differential equation

$$\nabla^2 T_m^* - \frac{\partial T_m^*}{\partial t^*} = \psi_m^* . \quad (6)$$

The heat flow per unit area at the earth's surface, F_m , is given by $K \partial T_m / \partial r$. If $F_m = F_m^0 F_m^*$, where F_m^0 is the characteristic heat flow per unit area, we have

$$F = F_m^0 F_m^* = \left(K \frac{T_m^0}{r^0} \right) \frac{\partial T_m^*}{\partial r^*} \Big|_{r^* = a^*}$$

or

$$F_m^* = \left(K \frac{T_m^0}{r^0 F_m^0} \right) \frac{\partial T_m^*}{\partial r^*} \Big|_{r^* = a^*} . \quad (7)$$

If $KT_m^0 / r^0 F_m^0 = 1$, we have

$$T_m^0 = r^0 F_m^0 / K . \quad (8)$$

Since we are not interested in the spherically symmetric distribution of either the temperature, T_s^* , the source density, ψ_s^* , or the heat flow, F_s^* , but are rather interested in the deviations from their spherical terms, let us set

$$\left. \begin{aligned} T_m^* &= T_s^* + T^* \\ F_m^* &= F_s^* + F^* \\ \psi_m^* &= \psi_s^* + \psi^* \end{aligned} \right\} , \quad (9)$$

where T^* , F^* , ψ^* are the deviations of temperature, heat flow, and source density from their spherical distributions. In view of the linearity of (6) we have finally

$$\nabla^2 T^* - \frac{\partial T^*}{\partial t^*} = \psi^* . \quad (10)$$

It is a reasonable assumption that the present temperature fluctuation, T^* , in the interior is controlled by a pattern of distribution of the radioactive elements, fixed since the crust and the mantle were consolidated. Assuming the age of the oldest rock (3.5 billion years) to be the age of the crust and the upper mantle, we can set $t = 0$ as the time at 3.5 billion years ago. At $t = 0$, when the pattern of distribution of the radioactive elements was just fixed, the fluctuation of temperature caused by this distribution is nil. The boundary condition of this problem is that $T^* = 0$ at the surface of the spherical earth, $r = a$, and T^* is bounded at the center of the earth, $r = 0$.

Using the method of the Laplace transform defined as

$$\bar{f}(\lambda) = \int_0^\infty f(x) e^{-\lambda x} dx ,$$

equation (10) becomes

$$\nabla^{*2} \bar{T}^* - \lambda \bar{T}^* = \frac{\psi^*}{\lambda} , \quad (11)$$

with boundary conditions

$$\bar{T}^* = 0 \text{ at } r^* = a^* .$$

T^* is bounded at $r^* = 0$. We have, from the differential equation (11),

$$\bar{T}^* = \iiint_V G(\vec{x}, \vec{x}'; \lambda) \frac{\psi^*(\vec{x}')}{\lambda} d^3x , \quad (12)$$

where G is the Green's function satisfying

$$\nabla^{*2} G - \lambda G = - \delta(\vec{x} - \vec{x}') \quad (13)$$

and the same boundary condition. $\delta(\vec{x} - \vec{x}')$ is the Dirac delta function. Assuming that $\varphi_{\ell, m, k}$ is the eigenfunction of the differential equation

$$\nabla^2 \varphi + \lambda \varphi = 0 ,$$

with the same boundary condition, corresponding to the eigenvalue $\lambda_{\ell, k}$, then G may be expressed as the bilinear expansion

$$G(\vec{x}, \vec{x}'; \lambda) = - \sum_{\ell, m, k=0}^{\infty} \frac{\varphi_{\ell, m, k}(\vec{x}) \varphi_{\ell, m, k}(\vec{x}')}{\lambda + \lambda_{\ell, k}} . \quad (14)$$

Hence

$$\bar{T}^*(\vec{x}; \lambda) = - \sum_{\ell, m, k=0}^{\infty} \frac{b_{\ell, m, k} \varphi_{\ell, m, k}(\vec{x})}{\lambda(\lambda + \lambda_{\ell, k})} , \quad (15)$$

where

$$b_{\ell, m, k} = \iiint_V \varphi_{\ell, m, k}(\vec{x}') \psi^*(\vec{x}') d^3x . \quad (16)$$

Thus

$$T^*(\vec{x}; t) = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} -e^{\lambda t} \sum_{\ell, m, k=0}^{\infty} \frac{b_{\ell, m, k} \varphi_{\ell, m, k}(\vec{x})}{\lambda(\lambda + \lambda_{\ell, k})} d\lambda . \quad (17)$$

Since $\lambda_{\ell, k}$ is a sequence of real, positive numbers, the integral has simple poles at $\lambda = 0$ and $\lambda = -\lambda_{\ell, k}$. The integral along the circular arc on the complex- λ plane, centering at the origin and lying to the left of the line of integration, tends to zero as the radius of the arc tends to infinity. Cauchy's theorem gives

$$T^*(\vec{x}; t) = - \sum_{\ell, m, k=0}^{\infty} \left[\frac{b_{\ell, m, k} \varphi_{\ell, m, k}(\vec{x})}{\lambda_{\ell, k}} - \frac{b_{\ell, m, k} \varphi_{\ell, m, k}(\vec{x})}{\lambda_{\ell, k}} e^{-\lambda_{\ell, k} t} \right] \quad (18)$$

or, replacing $b_{\ell, m, k}$ by the integral expression, we have

$$T^*(\vec{x}, t) = - \iiint_V \sum_{\ell, m, k=0}^{\infty} \varphi_{\ell, m, k}(\vec{x}) \varphi_{\ell, m, k}(\vec{x}') \left(\frac{1 - e^{-\lambda_{\ell, k} t}}{\lambda_{\ell, k}} \right) \psi^*(\vec{x}') d^3x . \quad (19)$$

In the spherical coordinates, the eigenfunction $\varphi_{l,m,k}$, as is well known, is

$$N(l,m,k) j_l \left(\lambda_{l,k} \frac{r^*}{a^*} \right) P_l^m (\cos \theta) \frac{\cos m\varphi}{\sin m\varphi}, \quad (20)$$

where $N(l,m,k)$ is the normalizing factor and $\lambda_{l,k}$ is the k -th root of the l -th order spherical Bessel function of the first kind: j_l , defined as

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+\frac{1}{2}}(z).$$

The eigenfunction is bounded at $r^* = 0$ and is single-valued for every point in space. We may then write T^* as

$$\begin{aligned} T^*(r^*, \theta, \varphi, t) = & - \sum_{l,m,k=0}^{\infty} N^2(l,m,k) j_l \left(\lambda_{l,k} \frac{r^*}{a^*} \right) P_l^m (\cos \theta) \frac{\cos m\varphi}{\sin m\varphi} \left(\frac{1-e^{-\lambda_{l,k} t}}{\lambda_{l,k}} \right) \\ & \times \iiint_V j_l \left(\lambda_{l,k} \frac{r^{*'}}{a^*} \right) P_l^m (\cos \theta') \\ & \times \frac{\cos m\varphi'}{\sin m\varphi'} \psi^*(r^{*'}, \theta', \varphi') r^{*'}{}^2 \sin \theta' dr^{*'} d\theta' d\varphi', \end{aligned} \quad (21)$$

where

$$N(l,m,k) = \left[\frac{2(2l+1)(l-m)!}{r_a^{*3}(l+m)! j_l'^2(\lambda_{l,k})} \right]^{\frac{1}{2}}. \quad (22)$$

Formally, we can express $\psi^*(r^{*'}, \theta', \varphi')$ in the following expansion

$$\psi^*(r^{*'}, \theta', \varphi') = \sum_{u,v,w=0}^{\infty} j_u \left(\lambda_{u,w} \frac{r^{*'}}{a^*} \right) P_u^v (\cos \theta') \begin{cases} d_{u,v,w} \cos v\varphi' \\ e_{u,v,w} \sin v\varphi' \end{cases}; \quad (23)$$

we have, after replacing this in the expression of T^* (21) and integrating

$$T^*(r^*, \theta, \varphi; t^*) = - \sum_{\ell, m, k=0}^{\infty} j_{\ell} \left(\lambda_{\ell, k} \frac{r^*}{a^*} \right) P_{\ell}^m (\cos \theta) \times (d_{\ell, m, k} \cos m\varphi + e_{\ell, m, k} \sin m\varphi) \cdot \frac{1 - e^{-\lambda_{\ell, k} t^*}}{\lambda_{\ell, k}}. \quad (24)$$

Now

$$F^* = \left. \frac{\partial T^*}{\partial r^*} \right|_{r^* = a^*}, \quad (25)$$

so

$$F^*(\theta, \varphi; t^*) = - \sum_{\ell, m, k=0}^{\infty} j'_{\ell} (\lambda_{\ell, k}) P_{\ell}^m (\cos \theta) \times (d_{\ell, m, k} \cos m\varphi + e_{\ell, m, k} \sin m\varphi) \cdot \frac{1 - e^{-\lambda_{\ell, k} t^*}}{\lambda_{\ell, k}}. \quad (26)$$

For a certain time $t = t^0 = 3.5$ billion years, i.e., $t^* = 1$, we have

$$F^*(\theta, \varphi; 1) = - \sum_{\ell, m, k=0}^{\infty} j'_{\ell} (\lambda_{\ell, k}) \left(\frac{1 - e^{-\lambda_{\ell, k}}}{\lambda_{\ell, k}} \right) \cdot P_{\ell}^m (\cos \theta) (d_{\ell, m, k} \cos m\varphi + e_{\ell, m, k} \sin m\varphi). \quad (27)$$

Now, from the analysis of the surface heat-flow distribution, $F^*(\theta, \varphi; 1)$ may be expressed in the expansion of spherical harmonics, that is,

$$F^*(\theta, \varphi; 1) = - \sum_{\ell, m=0}^{\infty} P_{\ell}^m (\cos \theta) (f_{\ell, m}^* \cos m\varphi + g_{\ell, m}^* \sin m\varphi). \quad (28)$$

By comparing the corresponding coefficients in the above two expressions for F^* , we arrive at

$$f_{lm}^* = - \sum_{k=0}^{\infty} j'_l(\lambda_{lk}) \left(\frac{1-e^{-\lambda_{lk}}}{\lambda_{lk}} \right) d_{l,m,k} , \quad (29)$$

and

$$g_{lm}^* = - \sum_{k=0}^{\infty} j'_l(\lambda_{lk}) \left(\frac{1-e^{-\lambda_{lk}}}{\lambda_{lk}} \right) e_{l,m,k} . \quad (30)$$

From the above two equations we see that for a given $f_{l,m}$ and $g_{l,m}$, the solutions for the corresponding $d_{l,m,k}$ and $e_{l,m,k}$ are not uniquely determined. This situation is what can be expected from our physical intuition that for a given surface heat flow resulting from heat sources under the surface, there are infinite ways that the corresponding heat sources may distribute. We have, therefore, to impose a stricter assumption on the problem before we can get an answer. For example, we could assume that the source density ψ^* is not a function of r^* , or that it is directly proportional to r^* or to r^{*2} , etc. Since we are dealing only with the departures of the distribution of the heat sources from the main spherical term, these considerations may not be far from reasonable.

Model I. ψ^* is a function independent of r .

$$\text{Set} \quad \psi^* = \sum_{l=0}^{\infty} \sum_{m=0}^l \left(p_{l,m}^{(1)} \cos m\varphi + q_{l,m}^{(1)} \sin m\varphi \right) P_l^m(\cos \theta) . \quad (31)$$

Since ψ^* can also be expanded in the form of (24), by comparing equations (24) and (31), we have

$$\sum_{k=0}^{\infty} d_{l,m,k} j_l\left(\lambda_{l,k} \frac{r^*}{a^*}\right) = p_{l,m}^{(1)} , \quad (32)$$

and

$$\sum_{k=0}^{\infty} e_{l,m,k} j_l\left(\lambda_{l,k} \frac{r^*}{a^*}\right) = q_{l,m}^{(1)} . \quad (33)$$

The functions $j_\ell(\lambda x)$ are solutions for the following differential equation in the Sturm-Liouville form

$$(x^2 j'_\ell)' + [\lambda^2 x^2 - \ell(\ell+1)] j_\ell = 0$$

and have roots $\lambda_{\ell,k}$ such that $j_\ell(\lambda_{\ell,k}) = 0$; hence they form an orthogonal set of functions on the interval $0 < x < 1$, with respect to the weight function $p(x) = x^2$. If $x = r^*/a^*$, (32) and (33) then give

$$d_{\ell,m,k} = \frac{2p_{\ell,m}^{(1)}}{[j'_\ell(\lambda_{\ell,k})]^2} \int_0^1 x^2 j_\ell(\lambda_{\ell,k} x) dx \quad (34)$$

$$e_{\ell,m,k} = \frac{2q_{\ell,m}^{(1)}}{[j'_\ell(\lambda_{\ell,k})]^2} \int_0^1 x^2 j_\ell(\lambda_{\ell,k} x) dx. \quad (35)$$

Substituting (34) and (35) into (29) and (30), respectively, we get

$$\left. \begin{matrix} f_{\ell,m}^* \\ g_{\ell,m}^* \end{matrix} \right\} = \left. \begin{matrix} 2p_{\ell,m}^{(1)} \\ 2q_{\ell,m}^{(2)} \end{matrix} \right\} \left[+ \sum_{k=0}^{\infty} \frac{(1 - e^{-\lambda_{\ell,k}})}{a^* j'_\ell(\lambda_{\ell,k})} \times \int_0^1 x^2 j_\ell(\lambda_{\ell,k} x) dx \right]. \quad (36)$$

$p_{l,m}$ and $q_{l,m}$ are then given by

$$\left. \begin{matrix} p_{l,m}^{(1)} \\ q_{l,m}^{(2)} \end{matrix} \right\} = \left. \begin{matrix} f_{l,m}^* \\ g_{l,m}^* \end{matrix} \right\} \left[+ \sum_{k=0}^{\infty} \frac{2(1 - e^{-\lambda_{l,k}})}{a^* j_{\ell}'(\lambda_{l,k})} \times \int_0^1 x^2 j_{\ell}(\lambda_{l,k} x) dx \right]^{-1}. \quad (37)$$

The temperature resulting from this distribution of heat sources is given by (24), in which $d_{l,m,k}$ and $e_{l,m,k}$ are given by (34) and (35).

In general, let us write

$$\psi^* = \sum_{l=0}^{\infty} \sum_{m=0}^l R(r^*) P_{\ell}^m(\cos \theta) (p_{l,m} \cos m\varphi + q_{l,m} \sin m\varphi), \quad (38)$$

where $R(r^*)$ is any function of r^* . Following the same procedure as for Model I, we have

$$\left. \begin{matrix} p_{l,m} \\ q_{l,m} \end{matrix} \right\} = \left. \begin{matrix} f_{l,m}^* \\ g_{l,m}^* \end{matrix} \right\} \left[\sum_{k=0}^{\infty} \frac{2(1 - e^{-\lambda_{l,k}})}{a^* j_{\ell}'(\lambda_{l,k})} \times \int_0^1 R(r^*) x^2 j_{\ell}(\lambda_{l,k} x) dx \right]^{-1}, \quad (39)$$

and

$$\left. \begin{matrix} d_{l,m,k} \\ e_{l,m,k} \end{matrix} \right\} = \left. \begin{matrix} p_{l,m} \\ q_{l,m} \end{matrix} \right\} \frac{2}{[j'_l(\lambda_{l,k})]^2} \int_0^1 R(r^*) x^2 j_l(\lambda_{l,k} x) dx . \quad (40)$$

Equations (39) and (40) can be solved either analytically or numerically, depending upon the character of the function $R(r^*)$. Some simple models are proposed as follows:

Model II. $R(r^*) = \delta(r^*)$,

where $\delta(r^*) = 1$ within a certain spherical layer in the earth, but is equal to zero elsewhere.

Model III. $R(r^*) = x$.

Model IV. $R(r^*) = x^2$.

For all the above models, analytical solutions are found for equations (39) and (40). Numerical results, to be given in a later paper, would indicate which one of these models is most realistic, or how a more realistic model can be proposed.

4. Surface-density anomaly and geoid height

If we write a surface-density anomaly σ to represent the accumulation of the density anomaly resulting from the uneven thermal expansion, we have

$$\sigma = - \int_0^1 \alpha T \rho dx , \quad (41)$$

where α is the coefficient of volume thermal expansion.

This surface-density anomaly, expressed in the expansion of spherical harmonics, can also be written as

$$\sigma = - \sum_{l=0}^{\infty} \sum_{m=0}^n (2l+1) (c_{l,m} \cos m\varphi + s_{l,m} \sin m\varphi) P_l^m(\cos \theta) . \quad (42)$$

By comparing (2), (24), (41), and (42), $c_{l,m}$ and $s_{l,m}$ are solved in terms of $d_{l,m,k}$ and $e_{l,m,k}$:

$$\left. \begin{matrix} c_{l,m} \\ s_{l,m} \end{matrix} \right\} = \frac{T_m^0}{2l+1} \sum_{k=0}^{\infty} \left. \begin{matrix} d_{l,m,k} \\ e_{l,m,k} \end{matrix} \right\} \frac{1 - e^{-\lambda_{l,k}}}{\lambda_{l,k}} \int_0^1 \alpha_p j_l(\lambda_{l,k} x) dx . \quad (43)$$

The external potential resulting from the surface-density anomaly expressed in (42) is (Jeffreys, 1959)

$$U = 4\pi a G' \sum_{l=0}^{\infty} \sum_{m=0}^l \left(\frac{a}{r}\right)^{l+1} (c_{l,m} \cos m\varphi + s_{l,m} \sin m\varphi) P_l^m(\cos \theta), \quad (44)$$

where G' is the gravitational constant. The geoid height corresponding to the above external potential is (Jeffreys, 1959)

$$g.h. = \frac{4\pi a^3}{M} \sum_{l=0}^{\infty} \sum_{m=0}^l (c_{l,m} \cos m\varphi + s_{l,m} \sin m\varphi) P_l^m(\cos \theta) , \quad (45)$$

where M is the mass of the earth. Or,

$$g.h. = \frac{3}{\bar{\rho}} \sum_{l=0}^{\infty} \sum_{m=0}^l (c_{l,m} \cos m\varphi + s_{l,m} \sin m\varphi) P_l^m(\cos \theta) , \quad (46)$$

where $\bar{\rho}$ is the mean density of the earth. If the geoid height so computed has the same order of magnitude as that in figure 1, then the assumption that the second-order variation in the gravitational field results from inhomogeneous distribution of radioactive elements in the interior is good.

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